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A NOTE ON THE RANGE OF COMPACT MULTIPLIERS OF MIXED-NORM SEQUENCE SPACE

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Abstract. In this note we consider the range as a range space of compact multipliers of mixed norm sequence spaces $l^{p,q}$, $0 \leq p, q \leq \infty$. In contrast to the general case we show that a compact multiplier always remain compact under reduction of the final space to the range space of multipliers.

Let us recall that a complex sequence $\{\lambda_n\}$ is of class $l^{p,q}$, $0 < p, q \leq \infty$, if

$$(1) \qquad \sum_{m=0}^{\infty} \Big(\sum_{n \in I(m)} |\lambda_n|^p \Big)^{q/p} < \infty,$$

where $I(0) = \{0\}$ and $I(m) = \{n \in \mathbb{N} : 2^{m-1} \leq n < 2^m\}$, for $m > 0$. In the case where p or q is infinite, replace the corresponding sum by a supremum. It is known that $l^{p,q}$ with norm

$$(2) \qquad \|\lambda\| = \Bigg(\sum_{m=0}^{\infty} \Big(\sum_{n \in I(m)} |\lambda_n|^p \Big)^{q/p} \Bigg)^{1/q}, \quad (1 \leq p, q < \infty),$$

is a Banach space. Note that $l^{p,p} = l^p$, and that if p or q is infinite then the corresponding sum should be replaced by a supremum: thus

$$(3) \qquad \|\lambda\| = \sup_m \Big(\sum_{n \in I(m)} |\lambda_n|^p \Big)^{1/p}, \quad (1 \leq p < \infty,\ q = \infty).$$

Define

$$(4) \qquad \|\lambda\| = \sum_{m=0}^{\infty} \Big(\sum_{n \in I(m)} |\lambda_n|^p \Big)^{q/p}, \quad (1 \leq p < \infty,\ q < 1),$$

$$(5) \qquad \|\lambda\| = \sum_{m=0}^{\infty} \Big(\sup_{n \in I(m)} |\lambda_n| \Big)^{q}, \quad (p = \infty,\ q < 1).$$

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$$(6)\qquad \|\lambda\| = \sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{q/p}, \quad (p<1,\, q\le p),$$

$$(7)\qquad \|\lambda\| = \Big(\sum_{m=0}^{\infty}\Big(\sum_{n\in I(m)}|\lambda_n|^p\Big)^{q/p}\Big)^{q/p}, \quad (p<1,\, p\le q<\infty),$$

and

$$(8)\qquad \|\lambda\| = \sup_m \sum_{n\in I(m)}|\lambda_n|^p, \quad (p<1,\, q=\infty).$$

For economy the dependence of $\|\lambda\|$ on p and q has not been indicated but it should be borne in mind. Thus in the case $1\le p,q\le\infty$, $(l^{p,q},\|\cdot\|)$ is a Banach space, usually called the mixed-norm space $l^{p,q}$, in the case $1\le p\le\infty$, $0<q<1$, and in the case $0<p<1$, $q\le p$, it is a complete q-normed space; finally, in the case $0<p<1$, $p\le q$, it is a complete p-normed space (see e.g. [6], [7]).

For any two subsets E and F of l^∞, the set of multipliers from E to F (denoted by (E,F)) is the set of all $\lambda=\{\lambda_n\}\in l^\infty$ such that $\lambda a=\{\lambda_n a_n\}$ is an element of F for all $a=\{a_n\}\in E$. Let $T_\lambda: E\mapsto F$ be the operator defined by $T_\lambda(a)=\lambda a$, $(a\in E)$. For the convenience of a reader, recall the following well-known theorem of Kellog [6, Theorem 1].

Theorem (Kellog) 1. *Let $1\le r,s,u,v\le\infty$, and define p and q by*

$$1/p=1/u-1/r \quad \text{if} \quad r>u, \qquad p=\infty \quad \text{if} \quad r\le u,$$
$$1/q=1/v-1/s \quad \text{if} \quad s>v, \qquad q=\infty \quad \text{if} \quad s\le v.$$

Then $(l^{r,s},l^{u,v})=l^{p,q}$.

Racall that Kellog (in [6, Theorem 1]) proved that the operator $T_\lambda:$ $: l^{r,s}\mapsto l^{u,v}$, defined by $T_\lambda(x)=\lambda x$, $(x\in l^{r,s})$, is a bounded linear operator and that its operator norm $\|T_\lambda\|$ is equal to $\|\lambda\|$. Let us remark that it was observed (see e.g. [1, Theorem 7.1, Theorem 8.1] or [3, Lemma 1.1.2]) that Kellog's theorem is true for $0<r,s,u,v\le\infty$. In [4], [5] we have investigated the Hausdorff measure of noncompactness (see e.g [8], [9]) of the operator T_λ, and prove necessary and sufficient conditions for T_λ to be compact. For the convenience of a reader, let us recall.

Theorem 2 [5, Corollary 2.3]. *Let $0<r,u\le\infty$, $0<s,v\le\infty$, and define p and q by*

$$1/p=1/u-1/r \quad \text{if} \quad r>u, \qquad p=\infty \quad \text{if} \quad r\le u,$$
$$1/q=1/v-1/s \quad \text{if} \quad s>v, \qquad q=\infty \quad \text{if} \quad s\le v.$$

Then, for $\lambda \in (l^{r,s}, l^{u,v}) = l^{p,q}$, *we have:*

(9) T_λ*is a compact, if* $v < s$,

(10) T_λ*is a compact* $\iff \limsup_{n\to\infty} |\lambda_n| = 0$, *if* $s \le v$ *and* $r \le u$,

(11) T_λ *is a compact* $\iff \limsup_{m\to\infty} \left(\sum_{n\in I(m)} |\lambda_n|^p \right)^{1/p} = 0$, *if* $s \le v$ *and* $r > u$.

The next theorem extends the result of Goldberg and Thorp [2, Theorem]. Goldberg and Thorp considered multiplier $T_\lambda : l_p \mapsto l_q$, $1 \le p, q \le \infty$.

Theorem 3. *Let* $T_\lambda : l^{r,s} \mapsto l^{u,v}$, $0 < r, s, u, v \le \infty$ *a compact multiplier. If* $R(T_\lambda)$ *is the range of* T_λ *and* $T_{0\lambda} : l^{r,s} \mapsto R(T_\lambda)$ *is the reduced operator corresponding to* T_λ, *then* $T_{0\lambda}$ *is also compact.*

Proof. It is enough to suppose that all λ_i are non-zero. Let $K = \{x \in l^{r,s} : \|x\| \le 1\}$ and $x^{(n)} = (x_i^{(n)}) \in K$, $i = 1, 2, \dots$. Suppose that $T_\lambda x^{(n)} \to y = (y_i) \in l^{u,v}$, $n \to \infty$. We have only to prove that $\{y_i/\lambda_i\} \in l^{r,s}$. Set $u_m = \{y_1/1, y_2/2, \dots, y_m/m, 0, 0, \dots\}$, $m = 1, 2, \dots$, and $h_m^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \dots \dots, x_m^{(n)}, 0, 0 \dots\}$, $m = 1, 2, \dots$, $n = 1, 2, \dots$. Clearly $u_m, h_m^{(n)} \in l^{r,s}$ and

(12) $$\|u_m\| \le \|u_m - h_m^{(n)}\| + \|h_m^{(n)}\| \le \|u_m - h_m^{(n)}\| + 1$$

Now, since $T_\lambda x^{(n)} \to y$, $n \to \infty$, by (12) (take $n \to \infty$) we get $\|u_m\| \le 1$, $m = 1, 2, \dots$. Hence $\|y_i/\lambda_i\|_{r,s} \le 1$, i.e., $(y_i/\lambda_i) \in l^{r,s}$. ∎

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